A Consumption-Based Asset Pricing Model with Accounting Numbers: Corporate Earnings and Book Value Are as Real as Consumption

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Abstract
Combining (i) a consumption-based asset pricing model without “market” return derived by Restoy and Weil (2004) and (ii) a log-linearized clean-surplus relation by Vuolteenaho (2000), I have developed a consumption and accounting-based asset pricing model. Accounting numbers are crucial ingredients as real as consumption for utility-maximizing rational asset pricing rather than pieces of information possibly useful for investment decision-making. The model is, however, directly linked to current valuation studies in the accounting literature, and can be considered a data-acceptable consumption-based Ohlson model.
Constraint maximization, after all, could be understood as the pursuit of net worth subject to the constraints of balance sheet and income statements.

Arjo Klamer and Deidre N. McCloskey

1. Introduction

Researchers on financial markets have an enormous amount of hard data, that is, records of real transactions. However, this seemingly blissful situation may be a curse. Summers (1985) criticizes that “Financial economists like ketchupal economists work only with hard data and are concerned with the interrelationships between the prices of different financial assets” satirically comparing financial economists with imagined researchers who only study the relative prices (only hard data!) among different ketchups in the market. In this paper I would like to take this criticism seriously and seek a consumption and accounting-based fundamental asset pricing model.

To be concrete, following two preceding research programs, (i) an application of non-traditional utility on asset pricing and (ii) a log-linear approximation of an accounting identity, I try to find an asset pricing model based on not only corporate profitability but also utility maximization.

As for (i) the asset pricing part, based on Kreps and Porteus (1978), Epstein and Zin (1989) and Weil (1990) have developed a generalized recursive utility specification, which decouples intertemporal substitution and risk aversion. This specification is as tractable as the conventional power utility one in deriving Euler equations for asset returns, which Epstein and Zin (1989) first obtain. Synthesizing this non-expected utility formulation, log-linear approximation of budget constraints based on Campbell and Shiller (1989), and variance decomposition with vector autoregression (VAR) based on Campbell (1991), Campbell (1993) has derived a consumption-based asset pricing model without consumption data. Extending this framework, Campbell and Vuolteenaho (2004) obtain a two-beta ICAPM model, and Campbell et al. (2005) a two-by-two model. Though the unobservable wealth portfolio is routinely (in the above mentioned studies also) equated with leading market proxy indices in

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1 See Backus, Routledge and Zin (2004) for this and other non-traditional utility specifications.
empirical research, there is no compelling reason to justify this practice\(^2\). On the other hand, based on the same non-expected utility framework, Restoy and Weil (2004) take a different course from Campbell (1993) and obtain a consumption-based asset pricing model without the wealth portfolio. I follow Restoy and Weil (2004) in asset pricing modeling.


Concerning related accounting literature, though such notable contributions as Easton and Monahan (2005) conduct empirical research based on the above mentioned log-linear clean-surplus relation, this paper seems novel in that it explicitly introduces consumption into the accounting-based valuation model under a (non-expected) utility maximizing framework. Needles to say, Ohlson (1995) has influenced the accounting literature profoundly. As is shown later, the model presented here can be interpreted as an extension of the residual income model.

After non-expected utility is introduced in Section 2, log-linear approximations of consumption and individual assets (portfolio) are derived in Sections 3 and 4 respectively. Then, based on asset pricing without “market” return in Section 5, we obtain a consumption-based asset pricing model with accounting numbers. Section 7 concludes the paper.

2. Epstein-Zin-Weil Utility

A utility-maximizing representative agent is assumed to live infinitely, but his optimization problem can be formulated as the one consisting of two dates, that is, today and the future. His future utility is indexed by the certainty equivalent\(^3\) of his expected future random utility \(U_{t+1}\) at time \(t\) with a (relative) risk aversion parameter \(\gamma\)

\[
\mu_t = \left( E_t \left[ \frac{U_{t+1}^{1-\gamma}}{1-\gamma} \right] \right)^{\frac{1}{\gamma}}.
\]

With an aggregator function

\(^2\) Although Campbell (1996) and other prominent authors incorporate labor income into asset pricing with simplifying assumptions, wealth is not limited to tradable financial assets and human capital, as pointed out by Restoy and Weil (2004).

\(^3\) A certainty equivalent for log-utility needs a distinct specification, which I omit for simplicity. See Weil (1990) for details.
the agent computes his utility at \( t \) weighing his current consumption and utility at \( t+1 \) in an certainty equivalent form with a CES-type aggregator

\[
B(C, \mu) = \left( C^{\frac{1-\eta}{\eta}} + \delta \mu^{\frac{1-\eta}{\eta}} \right)^{\frac{1}{1-\eta}},
\]
while \( \eta \) is a parameter for (elasticity of) intertemporal substitution distinct from one for risk aversion.

Then, the agent’s utility function at \( t \) is expressed as

\[
U_t = \left[ C_t^{\frac{1}{\eta}} + \delta \sum_{j=0}^{\infty} \delta^j C_{t+j} \right]^{\frac{1}{1-\eta}}.
\]

This specification is not of a different kind but a natural generalization of a conventional utility function used in the asset pricing literature. For example, if there is no uncertainty, the utility function becomes

\[
U_t(C_t, C_{t+1}, \ldots) = \left[ C_t^{\frac{1}{\eta}} + \delta \left( C_{t+1}^{\frac{1}{\eta}} + \delta \left( C_{t+2}^{\frac{1}{\eta}} + \ldots \right) \right) \right]^{\frac{1}{1-\eta}} = \left[ \sum_{j=0}^{\infty} \delta^j C_{t+j} \right]^{\frac{1}{1-\eta}},
\]
which is a familiar time-separable utility function in deterministic environments. This specification makes it clear that \( \eta \) is a parameter for intertemporal substitution because risk aversion is moot under no uncertainty. If \( \frac{1}{\eta} = \gamma \), on the other hand, the function becomes

\[
U_t(C_t, C_{t+1}, \ldots) = \left[ C_t^{1-\gamma} + \delta E_t \left( C_{t+1}^{1-\gamma} + \delta E_{t+2} \left( C_{t+2}^{1-\gamma} + \ldots \right) \right) \right]^{\frac{1}{1-\gamma}} = \left\{ E_t \left[ \sum_{j=0}^{\infty} \delta^j C_{t+j} \right] \right\}^{\frac{1}{1-\gamma}},
\]
which is an usual time-separable utility function in stochastic environments. As is well-known, under a conventional power utility specification, risk aversion and intertemporal substitution parameters are the same. But there is no compelling reason to assume they really are.

The agent maximizes the utility of (2.1) under a budget constraint (law of motion). There are \( n \) assets and each asset is priced \( P_t \) paying dividends \( D_t \) at the end of \( t \). If the agent holds \( z_t \) shares of each asset, his wealth portfolio \( W_t \) at the end of time \( t \) becomes

\[
W_t = (D_t + P_t)'z_t,
\]
where
\[ D_i = \begin{pmatrix} D_{it} \\ \vdots \\ D_{mt} \end{pmatrix}, \quad P_i = \begin{pmatrix} P_{it} \\ \vdots \\ P_{mt} \end{pmatrix}, \quad z_i = \begin{pmatrix} z_{it} \\ \vdots \\ z_{mt} \end{pmatrix}. \]

In equilibrium, dividends must equal consumption. Therefore,
\[ C_i = \hat{1}'D_i = \sum_{t=1}^{\infty} D_{it}, \]
where
\[ \hat{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \]

I would like to emphasize that dividends include income from labor and physical property such as land because wealth, whether tangible or intangible, is not limited to financial assets.

Under this specification, Epstein and Zin (1989) have shown that the Euler equations for the wealth portfolio and individual assets becomes respectively\(^4\)
\[ E_i \left[ \delta \left( \frac{C_{t+1}}{C_i} \right)^{\frac{1}{\eta}} R_{t+1}^W \right] = 1, \quad (2.2) \]
\[ E_i \left[ \delta^\theta \left( \frac{C_{t+1}}{C_i} \right)^{\frac{\theta}{\eta}} (R_{t+1}^W)^{\theta-1} R_{t+1} \right] = 1. \quad (2.3) \]

In the above Euler equations, another parameter is introduced
\[ \theta = \frac{1 - \gamma}{\frac{1}{\eta} - 1} = \frac{\eta(1 - \gamma)}{\eta - 1}, \]
which measures the divergence from conventional expected utility (\( \theta = 1 \) in the latter case).

3. Log-Linear Approximations: Wealth Portfolio and Consumption

Although linear specifications used in Ohlson (1995) and Feltham and Ohlson (1999) allow us to get exact solutions, these models are inconsistent with the non-linear nature of most economic series, as pointed out by Fukui (2002)\(^5\). On the other hand, log-linear

\(^4\) Epstein (1988) is a less rigorous but easy-to-follow exposition.

\(^5\) See Campbell et al. (1997, p. 258) for a textbook exposition on the point.
approximations are a powerful technique to offer us analytical solutions, though approximate, for non-linear models which reflect data characteristics appropriately. Therefore, following Restoy and Weil (2004), I log-linearize the wealth portfolio and consumption process based on Campbell and Shiller (1988).

Rearranging the law of motion for the wealth portfolio

\[ W_{t+1} = R_{t+1}^W (W_t - C_t), \]

we obtain

\[ R_{t+1}^W = \frac{W_{t+1}}{W_t} \frac{1}{1 - \frac{C_t}{W_t}} = \frac{C_t}{W_t} \frac{1}{1 - \frac{C_t}{W_t}}. \]  

(3.1)

Converting consumption and wealth return into log forms

\[ c_t = \ln C_t, \quad r_{t+1} = \ln R_{t+1}^W, \]

we define a (log) consumption growth and a consumption-to-wealth ratio

\[ \Delta c_{t+1} \equiv c_{t+1} - c_t, \]
\[ a_t \equiv \ln \frac{C_t}{W_t}. \]

Then, the law of motion becomes

\[ r_{t+1} = \Delta c_{t+1} + a_t - a_{t+1} - \ln(1 - e^a). \]  

(3.2)

Using elementary results of calculus

\[ f(q) = \ln(1 - e^q), \]
\[ f'(q) = -\frac{e^q}{1 - e^q}, \]

we can log-linearize the law of motion around fixed \( a \) as follows:

\[ r_{t+1} \approx \Delta c_{t+1} + a_t - a_{t+1} - \left\{ \ln(1 - e^a) - \frac{e^a}{1 - e^a}(a_t - a) \right\} \]
\[ = \Delta c_{t+1} + \frac{1}{1 - e^a} \left( a_t - a_{t+1} - \left\{ \ln(1 - e^a) + \frac{e^a}{1 - e^a} a \right\} \right). \]

Defining an investment-to-wealth ratio at \( a \)

\[ \rho \equiv 1 - e^a = \frac{W - C}{W}, \]

we obtain a log-linearized law of motion
\[ r_{t+1}^w \approx \Delta c_{t+1} - a_{t+1} + \frac{1}{\rho} a_t - k, \]  

(3.3)

where

\[ k = \ln(1-e^a) + \frac{e^a}{1-e^a} a = \ln \rho + \frac{1-\rho}{\rho} \ln(1-\rho). \]

Taking expectations and differencing of (3.3) (\( \approx \) is replaced by \( = \) thereafter to simplify notations)

\[ (E_{t+1} - E_t)[r_{t+1}^w] = (E_{t+1} - E_t)[\Delta c_{t+1} - a_{t+1}], \]  

(3.4)

the conditional covariance of any asset return with the wealth portfolio is decomposed into those with the consumption and consumption-to-wealth ratio

\[ \sigma_{w,t} = \sigma_{r_c,t} - \sigma_{a,t}. \]  

(3.5)

Note that \( (E_{t+1} - E_t)[\cdot] \) operator\(^6\) plays a crucial role in the entire model building because it embodies news or surprise at \( t+1 \) relative to the information available at \( t \).

4. Log Linear Approximations: Individual Assets

It is often asserted that dividends are, after all, what investors/consumers ultimately care about and accounting numbers are just numbers full of distortion. On the other hand, when the financial press as well as corporate managers talk about corporate profitability, it is more on earnings, earnings-related ratios rather than on dividends. However, as long as the clean-surplus relation is maintained, dividends and earnings are interchangeable for valuation as shown by the residual income approach revived by Ohlson (1995). Moreover, as Black (1993) points out, earnings are a more appropriate measure for profitability than dividends, because the former try to capture value generation, while the latter merely record value distribution. It is true that accounting numbers are often manipulated for managers’ advantage, but most critically examined numbers by outsiders nonetheless. It is rather debatable that what we take as consumption data are less distorted than corporate earnings.

Anyway, no matter how much trust we put on the precision of accounting numbers as profitability indices, earnings \( I \) and book value \( B \) are directly linked to dividends \( D \) through the clean-surplus relation

\[ B_{t+1} = B_t + I_{t+1} - D_{t+1}. \]  

(4.1)

\(^{6}\) In Restoy and Weil (2004), it is called surprise operator and given a distinct notation \( S_{t+1} \).
Then, define an asset return and an ROE as:

\[
    r_{t+1} \equiv \ln R_{t+1} = \ln \frac{P_{t+1} + D_{t+1}}{P_t} = \ln \frac{D_{t+1} P_{t+1} + D_{t+1} D_t}{D_t P_t} \\
    = \ln \frac{D_{t+1}}{D_t} \cdot \frac{P_{t+1} + D_{t+1}}{D_t} \cdot \frac{1}{P_t + D_t} - 1
\]

(4.2)

\[
    roe_{t+1} \equiv \ln \frac{B_t + I_{t+1}}{B_t} = \ln \frac{B_{t+1} + D_{t+1}}{B_t} = \ln \frac{D_{t+1} B_{t+1} + D_{t+1} D_t}{D_t B_t} \\
    = \ln \frac{D_{t+1}}{D_t} \cdot \frac{B_{t+1} + D_{t+1}}{B_t} \cdot \frac{1}{D_t} - 1
\]

(4.3)

Converting dividends, a price and a book value into log forms

\[
    d_{t+1} = \ln D_{t+1}, \quad P_{t+1} = \ln P_{t+1}, \quad B_{t+1} = \ln B_{t+1}
\]

we define a (log) dividends growth and a price-to-dividends ratio, a book-to-dividends ratio and a price-to-book ratio (PBR)

\[
    \Delta d_{t+1} \equiv d_{t+1} - d_t, \\
    \phi_{t+1} \equiv \ln \frac{P_{t+1} + D_{t+1}}{D_{t+1}}, \\
    \omega_{t+1} \equiv \ln \frac{B_{t+1} + D_{t+1}}{D_{t+1}}, \\
    pbr_{t+1} \equiv \ln \frac{P_{t+1} + D_{t+1}}{B_{t+1} + D_{t+1}} = \ln \frac{P_{t+1} + D_{t+1}}{D_{t+1}} - \ln \frac{B_{t+1} + D_{t+1}}{D_{t+1}} = \phi_{t+1} - \omega_{t+1}.
\]

With these definitions, (4.2) and (4.3) are expressed as

\[
    r_{t+1} = \Delta d_{t+1} + \phi_{t+1} - \ln(e^{\phi} - 1), \quad (4.4) \]

\[
    roe_{t+1} = \Delta d_{t+1} + \omega_{t+1} - \ln(e^{\omega} - 1). \quad (4.5)
\]

Now I start to log-linearize the relation between asset return and accounting numbers (ratios) following Vuolteenaho (2000). Around some convex combination of the time series averages of the price-to-dividends and book-to-dividends ratios

\[
    \psi = \pi \phi + (1 - \pi) \omega,
\]

the following Taylor expansions are derived:

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7 In order to make exposition simplified, an i subscript is omitted.
\[
\ln(e^{\delta_{t+1}} - 1) \approx \ln(e^\nu - 1) + \frac{e^\nu}{e^\nu - 1}(\phi_{t+1} - \psi) = \ln(e^\nu - 1) - \frac{\psi e^\nu}{e^\nu - 1} + \frac{e^\nu}{e^\nu - 1}\phi_{t+1},
\]
\[
\ln(e^{\alpha_{t+1}} - 1) \approx \ln(e^\nu - 1) + \frac{e^\nu}{e^\nu - 1}(\omega_{t+1} - \psi) = \ln(e^\nu - 1) - \frac{\psi e^\nu}{e^\nu - 1} + \frac{e^\nu}{e^\nu - 1}\omega_{t+1}.
\]

Noticing the following relations
\[
\frac{e^{\delta_{t+1}}}{e^{\delta_{t+1}} - 1} = \frac{P_{t+1} + D_{t+1}}{P_{t+1}} = \frac{P_{t+1} + D_{t+1}}{D_{t+1}},
\]
\[
\frac{e^{\alpha_{t+1}}}{e^{\alpha_{t+1}} - 1} = \frac{B_{t+1} + D_{t+1}}{B_{t+1}} = \frac{B_{t+1} + D_{t+1}}{B_{t+1}},
\]
and defining a discount factor
\[
\rho \equiv \frac{e^\nu - 1}{e^\nu},
\]
(4.4) and (4.5) become
\[
r_{t+1} = \Delta d_{t+1} + \phi_{t+1} - \frac{1}{\rho}\phi_t - \chi, \tag{4.6}
\]
\[
roe_{t+1} = \Delta d_{t+1} + \omega_{t+1} - \frac{1}{\rho}\omega_t - \chi, \tag{4.7}
\]
respectively, where
\[
\chi = \ln(e^\nu - 1) - \frac{\psi e^\nu}{e^\nu - 1}.
\]
Combining (4.6) and (4.7)
\[
r_{t+1} = roe_{t+1} + \phi_{t+1} - \omega_{t+1} - \frac{1}{\rho}(\phi_t - \omega_t),
\]
we finally obtain
\[
r_{t+1} = roe_{t+1} + pbr_{t+1} - \frac{1}{\rho}pbr_t, \tag{4.8}
\]
which can be considered a log-linearized clean-surplus relation.

5. Asset Pricing without “Market” Return

Now we go back to the Euler equation for the wealth portfolio (2.3)
\[ E_t \left[ \delta^\theta \left( \frac{C_{t+1}}{C_t} \right)^{\vartheta} (R_{t+1}^W)^{\varphi-1} R_{j,t+1} \right] = 1. \]

Using (log) return variables instead of level variables, the equation can be expressed as

\[ E_t \left[ \delta^\theta \exp \left( -\frac{\theta \Delta c_{t+1}}{\eta} + (\theta - 1)r_{t+1}^W + r_{j,t+1} \right) \right] = 1. \]  

In order to derive a tractable asset pricing model, we make additional assumptions of lognormality and conditional homoscedasticity for asset returns and consumption growth. Then, the expected return of an asset at each time becomes

\[
E_t[r_{j,t+1}] = -\theta \ln \delta + \frac{\theta}{\eta} E_t[\Delta c_{t+1}] + (\theta - 1)E_t[r_{t+1}^W] \\
- \frac{1}{2} \left( \frac{\theta}{\eta} \right)^2 \sigma_{ic,t} + (\theta - 1)^2 \sigma_{WW,t} + \sigma_{ij,t} - \frac{2\theta(1-\theta)}{\eta} \sigma_{cW,t} - \frac{2\theta}{\eta} \sigma_{ic,t} + 2(\theta - 1)\sigma_{ij,t},
\]

while a risk-free return at each time becomes

\[
r_{F,t+1} = -\theta \ln \beta + \frac{\theta}{\eta} E_t[\Delta c_{t+1}] + (\theta - 1)E_t[r_{t+1}^M] \\
- \frac{1}{2} \left( \frac{\theta}{\eta} \right)^2 \sigma_{cc,t} + (\theta - 1)^2 \sigma_{WW,t} - \frac{2\theta(1-\theta)}{\eta} \sigma_{cW,t}. 
\]

Subtracting (5.2) from (5.3) and putting a Jensen’ inequality term into the left-side, we obtain a pricing formula based on the covariances with the wealth portfolio and consumption growth

\[
E_t[r_{j,t+1}] - r_{F,t+1} + \frac{1}{2} \sigma_{ij,t} = \frac{\theta}{\eta} \sigma_{ic,t} + (1-\theta)\sigma_{ij,t}. 
\]

Then, with (3.5), (5.4) becomes

\[
E_t[r_{j,t+1}] - r_{F,t+1} + \frac{1}{2} \sigma_{ij,t} = \frac{\theta}{\eta} \sigma_{ic,t} + (1-\theta)(\sigma_{ic,t} - \sigma_{ia,t}) \\
= \left( \frac{\theta}{\eta} + 1 - \theta \right) \sigma_{ic,t} + (\theta - 1)\sigma_{ia,t}. 
\]

Because the coefficient of the first term turns out to be \( \gamma \), we obtain

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8 Although the assumption of homoscedasticity seems more problematic than that of lognormality, changing variances are not so pronounced in low-frequency data as in high-frequency one, as claimed by Campbell and Vuolteenaho (2004). In addition, a GARCH specification, for example, does not materially affect the following results, as shown by Restoy and Weil (2004).
\[ E_t[r_{t+1}] - r_{F,t+1} + \frac{1}{2} \sigma_{u,t} = \gamma \sigma_{w,t} + (\theta - 1) \sigma_{u,t}, \]  

(5.5)

which is an asset pricing formula without market return.

Next, we seek a similar asset pricing formula for the wealth portfolio. To proceed as above,

\[
\theta \ln \delta - \frac{\theta}{\eta} E_t[\Delta c_{t+1}] + \theta E_t[r^w_{t+1}] + \frac{\theta^2}{2} \text{var}_t \left( r^w_{t+1} - \frac{\Delta c_{t+1}}{\eta} \right) = 0,
\]

and rearranging, we obtain

\[
E_t[r^w_{t+1}] = -\ln \delta + \frac{1}{\eta} E_t[\Delta c_{t+1}] - \frac{\theta}{2} \text{var}_t \left( r^w_{t+1} - \frac{\Delta c_{t+1}}{\eta} \right).
\]

(5.6)

Taking expectations of (3.3)

\[
E_t[r^w_{t+1}] = E_t \left[ \Delta c_{t+1} - a_{t+1} + \frac{1}{\rho} a_t - k \right],
\]

and rearranging, the consumption-to-wealth ratio becomes

\[
a_t = \rho E_t[r^w_{t+1} - \Delta c_{t+1} + a_{t+1} + k]
\]

\[
= \rho E_t \left[ -\ln \delta + \frac{1}{\eta} \Delta c_{t+1} - \frac{\theta}{2} \text{var}_t \left( r^w_{t+1} - \frac{\Delta c_{t+1}}{\eta} \right) - \Delta c_{t+1} + a_{t+1} + k \right]
\]

\[
= \rho \left[ -\ln \delta + k + E_t \left[ a_{t+1} - \left(1 - \frac{1}{\eta}\right) \Delta c_{t+1} \right] - \frac{\theta}{2} \text{var}_t \left[ a_{t+1} - \left(1 - \frac{1}{\eta}\right) \Delta c_{t+1} \right] \right].
\]

(5.7)

Using a transversality condition \( \lim_{j \to \infty} \rho^j a_{t+j} = 0 \), where

\[ z_j = a_t - \left(1 - \frac{1}{\eta}\right) \Delta c_t, \]

and iterating forward, we obtain

\[
a_t = \frac{\rho (-\ln \beta + k)}{1 - \rho} \left(1 - \frac{1}{\eta}\right) E_t \left[ \sum_{j=1}^{\infty} \rho^j \Delta c_{t+j} \right] - \frac{\theta}{2} E_t \left[ \sum_{j=1}^{\infty} \rho^j \text{var}_{t+j} \left[ z_{t+j} \right] \right]. \]

(5.8)

Defining

\[ h_{t+1} = E_t \left[ \sum_{j=1}^{\infty} \rho^j \Delta c_{t+1+j} \right], \]

(5.8) becomes
\begin{align}
a_i &= \left(1 - \frac{1}{\eta} \right) E_t \left[ \sum_{j=1}^{\infty} \rho^j \Delta c_{t+j} \right] \\
&\quad + \frac{\rho}{1 - \rho} \left[ -\ln \delta + k - \frac{\theta}{2} \left(1 - \frac{1}{\eta} \right)^2 \left( \sigma_{cc} + \sigma_{hh} + 2 \sigma_{ch} \right) \right]. 
\end{align}
(5.9)

Putting (5.9) into (3.3)

\begin{align}
r_{w}^{t+1} &= \Delta c_{t+1} - a_{t+1} + \frac{1}{\rho} a_i - k \\
&= \Delta c_{t+1} + \left(1 - \frac{1}{\eta} \right) E_t \left[ \sum_{j=0}^{\infty} \rho^j \Delta c_{t+j} \right] - \left(1 - \frac{1}{\eta} \right) \frac{1}{\rho} E_t \left[ \sum_{j=1}^{\infty} \rho^j \Delta c_{t+j} \right] \\
&\quad + \left[ -\ln \delta + k - \frac{\theta}{2} \left(1 - \frac{1}{\eta} \right)^2 \left( \sigma_{cc} + \sigma_{hh} + 2 \sigma_{ch} \right) \right],
\end{align}

and rearranging, we obtain

\begin{align}
r_{w}^{t+1} &= \frac{1}{\eta} \Delta c_{t+1} + \left(1 - \frac{1}{\eta} \right) \left( E_{t+1} - E_t \right) \left[ \sum_{j=0}^{\infty} \rho^j \Delta c_{t+j} \right] \\
&\quad + \left[ -\ln \delta + k - \frac{\theta}{2} \left(1 - \frac{1}{\eta} \right)^2 \left( \sigma_{cc} + \sigma_{hh} + 2 \sigma_{ch} \right) \right], 
\end{align}
(5.10)

Taking expectations of (5.10), the expected wealth portfolio return becomes

\begin{align}
E_t [r_{w}^{t+1}] &= \frac{1}{\eta} E_t [\Delta c_{t+1}] + \left[ -\ln \beta + k - \frac{\theta}{2} \left(1 - \frac{1}{\eta} \right)^2 \left( \sigma_{cc} + \sigma_{hh} + 2 \sigma_{ch} \right) \right]. 
\end{align}
(5.11)

At this point we are prepared to get rid of the unobservable wealth portfolio from asset pricing. Following Restoy and Weil (2004), we part company with Campbell (1993), who instead derives asset pricing without consumption by equating the wealth return with the conventional market return.

Taking expectations of and differencing (5.9), we obtain

\begin{align}
\left( E_{t+1} - E_t \right) [a_{t+1}] &= \left(1 - \frac{1}{\eta} \right) \left( E_{t+1} - E_t \right) \left[ \sum_{j=1}^{\infty} \rho^j \Delta c_{t+j} \right],
\end{align}

and consequently

\begin{align}
\sigma_{h} &= \left(1 - \frac{1}{\eta} \right) \sigma_{h}. 
\end{align}
(5.12)

Putting (5.12) into (5.1)
\[ E_t[r_{i,t+1}] - r_{F,t+1} + \frac{1}{2} \sigma_{ii} = \gamma \sigma_{ic} + (\theta - 1) \sigma_{ih}, \]
\[ = \gamma \sigma_{ic} - (\theta - 1) \left(1 - \frac{1}{\eta}\right) \sigma_{ih}, \]

we obtain the following asset pricing formula without wealth portfolio:
\[ E_t[r_{i,t+1}] - r_{F,t+1} + \frac{1}{2} \sigma_{ii} = \gamma \sigma_{ic} + \left(\frac{\gamma}{\eta} - 1\right) \sigma_{ih}. \] (5.13)

As for the wealth portfolio, taking expectations of and differencing the wealth portfolio version of (5.10)
\[ (E_{t+1} - E_t)r_{i,t}^W = \frac{1}{\eta} (E_{t+1} - E_t) \Delta c_{i,t} + \left(1 - \frac{1}{\eta}\right) (E_{t+1} - E_t) \left[ \sum_{j=0}^{\infty} \rho^j \Delta c_{i+1,t+j} \right], \]
and rearranging, we obtain
\[ (E_{t+1} - E_t)r_{i,t}^W = (E_{t+1} - E_t) \Delta c_{i,t} + \left(1 - \frac{1}{\eta}\right) (E_{t+1} - E_t) \left[ \sum_{j=1}^{\infty} \rho^j \Delta c_{i+1,t+j} \right]. \] (5.14)

Now the covariances with the wealth portfolio can be expressed without it
\[ \sigma_{ci} = \sigma_{cc} + \left(1 - \frac{1}{\eta}\right) \sigma_{ch}, \]
\[ \sigma_{hi} = \sigma_{ch} + \left(1 - \frac{1}{\eta}\right) \sigma_{hh}. \] (5.15)

Because (5.13) holds for any asset or portfolio, it must hold for the wealth portfolio. Putting (5.15) into (5.13) with \( i = W \), we obtain
\[ E_t[r_{i,t}^W] - r_{F,t+1} + \frac{1}{2} \sigma_{WW} = \gamma \sigma_{cW} + \left(\frac{\gamma}{\eta} - 1\right) \sigma_{hW} \]
\[ = \gamma \left[ \sigma_{cc} + \left(1 - \frac{1}{\eta}\right) \sigma_{ch} \right] + \left(\frac{\gamma}{\eta} - 1\right) \left[ \sigma_{ch} + \left(1 - \frac{1}{\eta}\right) \sigma_{hh} \right]. \] (5.16)

Though the wealth portfolio is unobservable, its return should obey the above asset pricing equation if actual consumption is a realization on the economy’s equilibrium path.

6. Consumption-Based Asset Pricing Model with Accounting Numbers

Define the risk premium of an asset relative to the wealth portfolio
\[ \pi_t^W = E_t[r_{i,t+1}^W] - E_t[r_{i,t+1}], \] (6.1)
which is constant because we assume conditional homoscedasticity. Then taking expectations
of (4.8)

\[ E_t[r_{t+1}] = E_t \left[ \text{roe}_{t+1} + pbr_{t+1} - \frac{1}{\rho_i} pbr_t \right], \tag{6.2} \]

and rearranging, we obtain

\[ pbr_t = \rho_t E_t [\text{roe}_{t+1} - r_{t+1} + pbr_{t+1}] = \rho_t E_t [\text{roe}_{t+1} - r^w_{t+1} - \pi^w_t + pbr_{t+1}] \]. \tag{6.3}

With a transversality condition \( \lim_{j \to \infty} \rho^j_i pbr_{t+j} = 0 \), iterating forward, (6.3) becomes

\[ pbr_t = E_t \left[ \sum_{j=1}^{\infty} \rho^j_i (\text{roe}_{t+j} - r^w_{t+j}) \right] + \frac{\rho_i \pi^w_t}{1 - \rho_i}. \tag{6.4} \]

Taking expectations of and differencing (6.4), we obtain

\[ (E_{t+1} - E_t)[pbr_{t+1}] = (E_{t+1} - E_t) \left[ \sum_{j=1}^{\infty} \rho^j_i (\text{roe}_{t+j} - r^w_{t+j}) \right]. \tag{6.5} \]

Modifying the time subscripts of (5.10)

\[ r^w_{t+1,j} = \frac{1}{\eta} \Delta c_{t+j} + \left( 1 - \frac{1}{\eta} \right) \left( E_{t+1,j} - E_{t,j} \right) \left[ \sum_{j=0}^{\infty} \rho^j \Delta c_{t+j+k} \right] \]

\[ - \ln \beta + k - \frac{\theta}{2} \left( 1 - \frac{1}{\eta} \right)^2 \left( \sigma_{\epsilon c} + \sigma_{\epsilon h} + 2 \sigma_{\epsilon k} \right), \]

and taking expectations of and differencing between \( t+1 \) and \( t \), we obtain

\[ (E_{t+1} - E_t)[r^w_{t+1,j}] = \frac{1}{\eta} (E_{t+1} - E_t) \left[ \Delta c_{t+1,j} \right]. \tag{6.6} \]

With (6.6), (6.5) becomes

\[ (E_{t+1} - E_t)[pbr_{t+1}] = (E_{t+1} - E_t) \left[ \sum_{j=1}^{\infty} \rho^j_i (\text{roe}_{t+j} - r^w_{t+j}) \right]. \tag{6.7} \]

Taking expectations of and differencing (6.2)

\[ (E_{t+1} - E_t)[r_{t+1}] = (E_{t+1} - E_t)[\text{roe}_{t+1}] + (E_{t+1} - E_t)[pbr_{t+1}], \tag{6.8} \]

and combining (6.7) and (6.8), we obtain

\[ (E_{t+1} - E_t)[r_{t+1}] = (E_{t+1} - E_t)[\text{roe}_{t+1}] \]

\[ + (E_{t+1} - E_t) \left[ \sum_{j=1}^{\infty} \rho^j_i \text{roe}_{j+1} \right] - \frac{1}{\eta} (E_{t+1} - E_t) \left[ \sum_{j=1}^{\infty} \rho^j_i \Delta c_{t+j} \right], \tag{6.8} \]

which decomposes the return surprise into the current profitability, discounted future profitability and future consumption growth surprises.
Defining the future profitability and consumption growth both discounted by a factor specific to each asset $\rho_i$\footnote{Because $\rho_i$ is generally different from $\rho$ used for $h_{t+1}$, $h_{i,t+1}$ and $h_{t+1}$ are also distinct.},

\begin{align*}
  f_{i,t+1} &= E_{t+1} \left[ \sum_{j=1}^{\infty} \rho_i^j \text{roe}_{i,t+1+j} \right], \\
  h_{i,t+1} &= E_{t+1} \left[ \sum_{j=1}^{\infty} \rho_i^j \Delta c_{i,t+1+j} \right],
\end{align*}

(6.9)

we obtain the following relations between covariances:

\begin{align*}
  \sigma_{hc} &= \sigma_{\text{roe},c} + \sigma_{f,c} - \frac{1}{\eta} \sigma_{h,c}, \\
  \sigma_{hh} &= \sigma_{\text{roe},h} + \sigma_{f,h} - \frac{1}{\eta} \sigma_{h,h}.
\end{align*}

(6.10)

Putting (6.10) into (5.13), we finally obtain

\begin{align*}
  E_{t} \left[ r_{i,t+1} \right] - r_{F,t+1} + \frac{\sigma_{i,t}}{2} = \gamma \left( \sigma_{\text{roe},c} + \sigma_{f,c} - \frac{1}{\eta} \sigma_{h,c} \right) + \left( \gamma - \frac{1}{\eta} \right) \left( \sigma_{\text{roe},h} + \sigma_{f,h} - \frac{1}{\eta} \sigma_{h,h} \right),
\end{align*}

(6.11)

which is an asset pricing model solely based on consumption and accounting fundamentals.

This asset pricing formula can be expressed in beta formats. Define betas as

\begin{align*}
  \beta_{i,c} &= \frac{\sigma_{i,c}}{\sigma_{cc}}, \quad \beta_{\text{roe},c} = \frac{\sigma_{\text{roe},c}}{\sigma_{cc}}, \quad \beta_{f,c} = \frac{\sigma_{f,c}}{\sigma_{cc}}, \quad \beta_{h,c} = \frac{\sigma_{h,c}}{\sigma_{cc}}, \\
  \beta_{i,h} &= \frac{\sigma_{i,h}}{\sigma_{hh}}, \quad \beta_{\text{roe},h} = \frac{\sigma_{\text{roe},h}}{\sigma_{hh}}, \quad \beta_{f,h} = \frac{\sigma_{f,h}}{\sigma_{hh}}, \quad \beta_{h,h} = \frac{\sigma_{h,h}}{\sigma_{hh}}.
\end{align*}

Then, the two-beta and two-by-three-beta formulas are respectively

\begin{align*}
  E_{t} \left[ r_{i,t+1} \right] - r_{F,t+1} + \frac{\sigma_{i,t}}{2} &= \gamma \beta_{i,c} \sigma_{cc} + \left( \gamma - \frac{1}{\eta} \right) \beta_{i,h} \sigma_{hh} \\
  &= \gamma \left[ \beta_{\text{roe},c} + \beta_{f,c} - \frac{1}{\eta} \beta_{h,c} \right] \sigma_{cc} + \left( \gamma - \frac{1}{\eta} \right) \left[ \beta_{\text{roe},h} + \beta_{f,h} - \frac{1}{\eta} \beta_{h,h} \right] \sigma_{hh}.
\end{align*}

(6.12)

(6.13)

It is easy to see the following relations hold:

\begin{align*}
  \beta_{i,c} &= \beta_{\text{roe},c} + \beta_{f,c} - \frac{1}{\eta} \beta_{h,c}, \\
  \beta_{i,h} &= \beta_{\text{roe},h} + \beta_{f,h} - \frac{1}{\eta} \beta_{h,h}.
\end{align*}

(6.14)
Redefining some betas

\[
\beta_{hi,c}' = -\frac{1}{\eta} \beta_{hi,c}, \quad \beta_{hi,h}' = -\frac{1}{\eta} \beta_{hi,h},
\]

we can simplify the relations between two betas and two-by-three betas to

\[
\beta_{i,c} = \beta_{roei,c} + \beta_{fi,c} + \beta_{hi,c}',
\]
\[
\beta_{i,h} = \beta_{roeh,h} + \beta_{gi,h} + \beta_{hi,h}'.
\]

For empirical investigation, we have to estimate surprise terms, \( h_{t+1}, f_{t+1} \) and \( h_{t,t+1} \), which are unobservable and to be reconstructed from time-series data. Following Campbell (1991), we use a VAR specification

\[
z_{t+1} = a + \Gamma z_t + u_{t+1}, \quad (6.15)
\]

where

\[
a = \begin{pmatrix}
a_1 \\
\vdots \\
a_k \\
\end{pmatrix}, \quad \Gamma = \begin{pmatrix}
g_{11} & \cdots & g_{1k} \\
\vdots & \ddots & \vdots \\
g_{k1} & \cdots & g_{kk} \\
\end{pmatrix}, \quad z_{t+1} = \begin{pmatrix}
\Delta c_{t+1} \\
r_{t+1} \\
roe_{t+1} \\
\end{pmatrix}.
\]

Because there is no particular restriction for variables in \( z \), we can include favorite accounting data as well as economy-wide data but the return of a target asset and its ROE as well the consumption growth should be included. To estimate \( h_{t+1} \), market proxy returns and its ROE may be used. First, define a surprise vector

\[
t_{t+1} = (E_{t+1} - E_t) \left[ \sum_{j=1}^\infty \rho^j z_{t+1-j} \right]. \quad (6.16)
\]

Then, iterating (6.15) forward

\[
E_t[z_{t+1}] = E_t[a + \Gamma z_t + u_{t+1}] = a + \Gamma z_t,
\]
\[
E_t[z_{t+2}] = E_t[a + \Gamma z_t + u_{t+2}] = a + \Gamma E_t[z_{t+1}] = a + \Gamma(a + \Gamma z_t) = (I + \Gamma) a + \Gamma^2 z_t,
\]
\[
\vdots
\]
\[
E_t[z_{t+1-j}] = \left( \sum_{k=0}^{j-1} \Gamma^k \right) a + \Gamma^{j+1} z_t, \quad (6.17)
\]

and putting (6.17) into (6.16)
we obtain
\[ t_{t+1} = \rho \Gamma [I - \rho \Gamma]^{-1} u_{t+1}, \quad (6.18) \]
the first few elements of which are the surprise terms to be used for computation of expected asset returns. We have finally completed our task to derive a consumption-based fundamental asset pricing model with accounting numbers ready for empirical investigation.

### 7. Conclusion

In this paper, combining (i) an asset pricing model without “market” return derived by Restoy and Weil (2004) and (ii) a log-linearized clean-surplus relation by Vuolteenaho (2000), I have developed a consumption and accounting-based asset pricing model.

Accounting numbers are crucial ingredients as real as consumption for utility-maximizing rational asset pricing rather than pieces of information possibly useful for investment decision-making.

The model is, however, directly linked to current valuation studies in the accounting literature. Rearranging (6.8), we obtain
\[
(E_{t+1} - E_t)[r_{t+1}] = (E_{t+1} - E_t)[\text{roe}_{t+1}]
+ (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho_j \left( \text{roe}_{t+1+j} - \frac{1}{\eta} \Delta c_{t+1+j} \right). \quad (7.1)
\]

With this expression, it becomes clear that the model in this paper can be considered an extension of the residual income model. The second term is news on the sum of discounted future abnormal returns with the growth of consumption (scaled by the elasticity of intertemporal substitution) being cost of capital. Different from the original Ohlson (1995) model, our log-linearized model is data-acceptable in the sense that the time-series properties of accounting and economic data are consistent with the proposed specification\(^{10}\). We may call the model here as a data-acceptable consumption-based Ohlson model.

\(^{10}\) See Spanos (1995) and Fukui (2002) for a methodological elaboration.
Moreover, when extracting news or surprise about future profitability and consumption growth with VAR, we can include our favorite accounting pieces of information in the system and may interpret the existence of value relevance if the inclusion of some accounting information enhances the explanatory power of the model. Value relevance studies may become more disciplined with this reasonably flexible but theoretically grounded framework.

Much maligned accounting numbers have historically significant predictive components on asset returns as shown by, for example, Davis et al. (2000) and Campbell et al. (2005). On the other hand, it is notorious that consumption-based models often perform miserably when confronting market data\textsuperscript{11}. However, this sober fact should not be used as a pretext for abandoning consumption-based models. Unless we are determined to discard a (neo-classical) economic way of thinking entirely, we must link asset returns to consumption in one way or another. Scientists are expected to improve measurement if data are noisy or inconclusive. Good natural scientists indeed seem to follow this dictum faithfully. As Prescott (1986) boldly asserts, it is measurement rather than theory that should be revised when data at hand are recalcitrant. Better or worse, accountants’ responsibilities in national and business income measurement seem greater than ever.

\textsuperscript{11} Campbell (2003) and Cochrane (2006) are comprehensive surveys on the subject.
References


