A Theory of Accounting Relativity:
Double-Entry Bookkeeping as a Transformation of Coordinates

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Abstract
The dividend discount model is a special case of the residual income model and the fundamental value of the firm is a scalar invariant under any accounting system with the clean-surplus. Under double-entry bookkeeping, any accounting system is relative concerning valuation in the sense that the fundamental value is a scalar invariant. A dividends stream or any accounting sequence has no a priori conceptual as well as empirical grounds to claim its primacy over others in valuation.
Success is relative: It is what we can make of the mess we have made of things.

T. S. Eliot

1. Accounting Numbers Are More than Decision-Useful

Since Ball and Brown (1968), empirical research has produced a countless amount of evidence on investment decision-usefulness of accounting data. In this line of research, decision-usefulness is understood as to what extent accounting data, net income and book value in particular, is correlated with stock price. In other words, the question is how relevant accounting data is compared to non-accounting rival information when explaining stock price movement. However, these correlation studies seem to lack solid theoretical underpinnings. Cynics may assert that what is done in the name of empirical research is not different from statistical astrology trying to find the correlation between the zodiac and personal characteristics.

In a sense the residual income concept reintroduced by Ohlson (1995) has relieved empirical researchers of the above criticism. Although the concept itself is not new¹, empirical researchers have received the residual income model as an anchor to give them a theoretical discipline when conducting empirical studies², because the discounted value of a residual income sequence plus the book value of the first period equals the discounted value of dividends, one of the most established valuation model in corporate finance.

However, the fact that the residual income model gives us the same estimate as the dividend discount model does not necessarily mean the primacy of the latter over the former. In this paper, I want to show the relativity of accounting measurement in valuation from the point of view that

¹ See, for example, Preinreich (1936) and Edward and Bell (1961, ch. 2).
² Considering the non-stationarity of accounting and market data, a log-linearized version derived by Vuolteenaho (2000) is more appropriate than the original version of Ohlson (1995).
the dividend discount model is included in the family of the residual income model.

Section 2 describes the residual income model under a more general setting than usually presented. Section 3 shows the relativity of accounting measurement in valuing the fundamental value of the firm. Section 4 concludes the paper.

2. Residual Income Model

We will adopt a finite time span between the first and last (liquidation) periods for valuing the firm. The finite time framework allows us to avoid a technical but thorny problem of the transversality condition. Moreover the terminal value problem would be almost a moot question in valuation practice, because discounting factors for distant future periods become negligible if the time span is long enough. The time series of flow and stock is described in Figure 1.

![Flow and Stock Diagram]

**Figure 1: Flow and Stock**

The only constraint for an accounting system is to maintain the clean-surplus relation after the first period. The expected cost of capital is assumed to be time-varying, which is consistent with a current practice in the finance literature. In order to make notations as simple as possible, we omit expectations operators.

First define variables as:

- $VE_0$: Fundamental Value of the Firm at the Beginning of the First Period
- $DV_i$: (Net) Dividends
Then, with the clean-surplus,

\[ BE_i = BE_{i-1} + NI_i - DV_i \]

or

\[ DV_i = BE_{i-1} + NI_i - BE_i. \]

If we add all the dividends paid out during the entire time span, the amount becomes

\[
DV_1 + DV_2 + \ldots + DV_{N-1} + DV_N
= BE_0 + NI_1 - BE_1 + BE_1 + NI_2 - BE_2 + \ldots + BE_{N-2} + NI_{N-1} - BE_{N-1} + BE_{N-1} + NI_N - BE_N
= BE_0 + NI_1 + NI_2 + \ldots NI_{N-1} + NI_N - BE_N.
\]

Assume the remaining assets are cashed in at the end of the last period, which means any discrepancy between the book value and cash amount should be reflected in the net income in the last period. Then,

\[ BE_N = 0 \]

and we obtain

\[
DV_1 + DV_2 + \ldots + DV_{N-1} + DV_N = BE_0 + NI_1 + NI_2 + \ldots NI_{N-1} + NI_N.
\]

This relation is what Sunder (1997, p. 67) calls the law of conservation of income. On the one hand, the amount of dividends equals that of net income (plus the book value of the first period) if aggregated from the entire periods, because accruals only allocate earnings among periods. On the other hand, the discounted value of net income (plus the book value of the first period) is usually different from the fundamental value

3 Both amounts may coincide accidentally under a certain accounting system.
\[ VE_0 = \sum_{i=1}^{N} \frac{DV_i}{(1 + r_i)} \neq BE_0 + \sum_{i=1}^{N} \frac{NI_i}{(1 + r_i)} , \]

because the time series pattern of flow affects the fundamental value, the most important scalar in valuation studies.

However, it is not difficult to construct an accounting-based sequence whose discounted value equals the fundamental value. Of course, this is the residual income.

Define the residual income as an amount in excess of accounting-based normal returns

\[ RI_i = NI_i - r_i BE_{i-1} . \]

Then, we obtain

\[ BE_i = BE_{i-1} + NI_i - DV_i = BE_{i-1} + r_i BE_{i-1} + RI_i - DV_i = (1 + r_i) BE_{i-1} + RI_i - DV_i \]

or

\[ DV_i = (1 + r_i) BE_{i-1} - BE_i + RI_i . \]

Therefore, the dividends sequence can be expressed as

\[
\begin{align*}
DV_1 &= (1 + r_1) BE_0 - BE_1 + RI_1 \\
DV_2 &= (1 + r_2) BE_1 - BE_2 + RI_2 \\
& \vdots \\
DV_{N-1} &= (1 + r_{N-1}) BE_{N-2} - BE_{N-1} + RI_{N-1} \\
& \text{and} \\
DV_N &= (1 + r_N) BE_{N-1} + RI_N
\end{align*}
\]

because

\[ BE_N = 0 . \]

Inserting this dividends sequence into the dividend discount model
\[ VE_0 = \sum_{i=0}^{N} \frac{DV_i}{\prod_{k=1}^{i} (1 + r_k)} \],

and continuing

\[ VE_0 = \frac{DV_0}{1 + r_1} + \frac{DV_2}{(1 + r_1)(1 + r_2)} + \dots + \frac{DV_{N-1}}{(1 + r_1) \cdots (1 + r_{N-1})} + \frac{DV_N}{(1 + r_1) \cdots (1 + r_N)} \]

\[ = \frac{(1 + r_1) \cdot BE_0 - BE_1 + RI_1}{1 + r_1} + \frac{(1 + r_2) \cdot BE_1 - BE_2 + RI_2}{(1 + r_2)(1 + r_1)} + \dots \]

\[ + \frac{(1 + r_{N-1}) \cdot BE_{N-2} - BE_{N-1} + RI_{N-1}}{(1 + r_{N-1}) \cdots (1 + r_{N-2})} + \frac{(1 + r_N) \cdot BE_{N-1} + RI_N}{(1 + r_N) \cdots (1 + r_{N-1})} \]

\[ = \left( BE_0 - \frac{BE_1}{1 + r_1} + \frac{RI_1}{1 + r_1} \right) + \left[ \frac{BE_1}{1 + r_1} - \frac{BE_2}{(1 + r_1)(1 + r_2)} + \frac{RI_2}{(1 + r_1)(1 + r_2)} \right] + \dots \]

\[ + \left[ \frac{BE_{N-2}}{(1 + r_1) \cdots (1 + r_{N-2})} - \frac{BE_{N-1}}{(1 + r_1) \cdots (1 + r_{N-1})} + \frac{RI_{N-1}}{(1 + r_1) \cdots (1 + r_{N-1})} \right] \]

\[ + \frac{BE_{N-1}}{(1 + r_1) \cdots (1 + r_{N-1})} + \frac{RI_N}{(1 + r_1) \cdots (1 + r_N)} \]

\[ = BE_0 + \frac{RI_1}{1 + r_1} + \frac{RI_2}{(1 + r_1)(1 + r_2)} + \dots + \frac{RI_{N-1}}{(1 + r_1) \cdots (1 + r_{N-1})} + \frac{RI_N}{(1 + r_1) \cdots (1 + r_N)}, \]

we obtain

\[ VE_0 = \sum_{i=0}^{N} \frac{DV_i}{\prod_{k=1}^{i} (1 + r_k)} = BE_0 + \sum_{i=0}^{N} \frac{RI_i}{\prod_{k=1}^{i} (1 + r_k)}. \]

As is well-known, the fundamental value of the firm can be expressed as the discounted value of a residual income sequence plus the first-period beginning book value. Any accounting system only\(^4\) constrained with the clean-surplus will do, no matter how much beginning book value at the first period the firm records. Under the clean-surplus, any accounting “distortion” is self-correcting. The reason why the residual income model holds true under any accounting system is that it is a definition rather than a valuation model. The original Ohlson (1995) model is

\(^4\) We need not consider the transversality condition because we assume a finite series.
a valuation model not because it adopts the residual income model but because it assumes particular time series properties.

If we want to estimate the fundamental value accruing to not only shareholders but also creditors, we should use: asset book value instead of equity book value; weighted average cost of capital (WACC) instead of equity cost of capital; and residual net operating profit after taxes instead of residual income. Subtracting the present value of debts from the calculated amount, we obtain the estimate of the fundamental equity value. This is now widely practiced in the name of EVA®.

Although the residual income model holds true under any accounting system, the time series properties of residual income are different each other and dependent on an adopted system. Under unbiased accounting systems (e.g., current value accounting), residual income is a proxy for economic super-normal profit and its expected value is zero. Accounting adjustments used in EVA® calculation may be understood as an exercise for making an existing accounting system closer to this “ideal” one.

Conversely, positive residual income does not necessarily reflect economic super-normal profit under biased accounting systems. For example, if the entire R & D expenditure must be expensed as currently forced, even large residual income may be simply a kind of accounting mirage, not reflect super-normal profit.

3. Accounting System as a Transformation of Coordinates

We may claim each accounting system records the same economic phenomena in a different but inter-related way. Therefore, dividends (or cash flows) are not more essential than accounting earnings and book value based on double-entry bookkeeping. Neither the true accounting system exists.
If we define discount factors as:

\[
\delta_0 = 1 \\
\delta_i = \frac{1}{(1+r_i)\cdots(1+r_N)}, \quad i = 1, \ldots, N,
\]

the valuation models can be expressed as:

\[
VE_0 = \frac{DV_1}{1+r_1} + \frac{DV_2}{(1+r_1)(1+r_2)} + \cdots + \frac{DV_{N-1}}{(1+r_1)\cdots(1+r_{N-1})} + \frac{DV_N}{(1+r_1)\cdots(1+r_N)} \\
= \delta_0 \cdot 0 + \delta_1 DV_1 + \delta_2 DV_2 + \cdots + \delta_{N-1} DV_{N-1} + \delta_N DV_N \\
= BE_0 + \frac{RI_1}{1+r_1} + \frac{RI_2}{(1+r_1)(1+r_2)} + \cdots + \frac{RI_{N-1}}{(1+r_1)\cdots(1+r_{N-1})} + \frac{RI_N}{(1+r_1)\cdots(1+r_N)} \\
= \delta_0 BE_0 + \delta_1 RI_1 + \delta_2 RI_2 + \cdots + \delta_{N-1} RI_{N-1} + \delta_N RI_N
\]

or

\[
VE_0 = \sum_{i=0}^{N} \delta_i RI_i.
\]

In other words, the firm value can be regarded as an inner product of discount factors and residual income\(^5\).

If we define vectors of discount factors, dividends and residual income as:

\[
\delta = \begin{pmatrix} \delta_0 \\ \delta_1 \\ \vdots \\ \delta_N \end{pmatrix}, \\
d = \begin{pmatrix} DV_1 \\ \vdots \\ DV_N \end{pmatrix}, \\
a = \begin{pmatrix} BE_0 \\ NI_1 - r_1 BE_0 \\ \vdots \\ NI_N - r_N BE_{N-1} \end{pmatrix}, \\
\delta = \begin{pmatrix} BE_0 \\ NI_1 - \frac{\delta_0 - \delta_1}{\delta_1} BE_0 \\ \vdots \\ NI_N - \frac{\delta_{N-1} - \delta_N}{\delta_N} BE_{N-1} \end{pmatrix}
\]

the firm value can be expressed as:

\[
VE_0 = \delta d = \delta a.
\]

This expression clearly shows that the residual income model make the firm value a scalar invariant under any accounting system, i.e., transformation of coordinates, with the clean-surplus

\(^5\) The book value at the beginning of the first period is interpreted as the residual income for the period 0.
relation.

With the newly defined discount factors as well as the clean-surplus

\[ BE_i = BE_{i-1} + NI_i - DV_i \]

and the zero terminal book value condition

\[ BE_N = 0, \]

the residual income and dividends can be expressed as:

\[ RI_i = NI_i - rBE_{i-1} = NI_i - \frac{\delta_{i+1} - \delta_i}{\delta_i} BE_{i-1} \]

\[ DV_i = rBE_{i-1} - BE_i + RI_i = \frac{\delta_{i+1}}{\delta_i} BE_{i-1} - BE_i + RI_i. \]

With the \((N+1)\)-dimensional vector and \((N+1) \times (N+1)\) matrices

\[
I = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}
\]

\[
\Delta = \begin{pmatrix} \delta_0 & 0 & \cdots & 0 \\ 0 & \delta_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_N \end{pmatrix}, \quad \tilde{\Delta} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \delta_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_{N-1} \end{pmatrix}
\]

we can express the \((N+1)\)-dimensional cost of capital vector as:
If we further define the \((N+1)\)-dimensional income and book value vectors as:

\[
e = \begin{pmatrix} 0 \\ NI_1 \\ \vdots \\ NI_N \end{pmatrix}, \quad b = \begin{pmatrix} BE_0 \\ BE_1 \\ \vdots \\ BE_N \end{pmatrix},
\]

the residual income vector can be expressed as:

\[
d = \begin{pmatrix} BE_0 \\ R\!I_1 \\ \vdots \\ R\!I_N \end{pmatrix} = \begin{pmatrix} NI_1 - r_1BE_0 \\ \vdots \\ NI_N - r_NBE_{N-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ NI_1 \\ \vdots \\ NI_N \end{pmatrix} - \begin{pmatrix} 0 \\ r_1BE_0 \\ \vdots \\ r_NBE_{N-1} \end{pmatrix} = Pb + e - r'Qb = (P - r'Q)b + e = \begin{pmatrix} P + t' \tilde{\Delta}^{-1} - Q \end{pmatrix}b + e.
\]

With the vectors and matrices defined above and the clean-surplus relation

\[
DV_i = BE_{i-1} + NI_i - BE_i,
\]

we can express the dividends vector as:

\[
d = (P + Q)b + e - b = (P + Q - I)b + e.
\]

As is clearly seen from this equation, though a given dividends sequence cannot pin down income and book value vectors, a definite dividends series can be derived once income and book value vectors are given. Therefore it may not be hyperbole to claim that net income and book value are more essential than dividends for valuation.

Although we tend to think the residual income model is derived from the dividend discount model because what investors care about is how much they receive in cash, we can go in reverse.
Actually, accounting transactions implicitly used in the dividend discount model are the simplest under the double-entry bookkeeping framework.

In concrete, when dividends are paid out, a flow transaction

Cash $DV / Income $DV

and a stock transaction

Dividends $DV / Cash $DV

are simultaneously recorded, while no other accounting transaction exists during the period. It is easy to see this accounting system satisfies the clean-surplus. Because cash to be paid during the period is the only stock account and paid out as income by definition, the book value at the end of period is always zero and net income equals dividends. Then, residual income equals dividends

$$RI_i = NI_i - r_iBE_{i-1} = DV_i,$$

and we obtain

$$VE_0 = BE_0 + \sum_{i=1}^{N} \frac{RI_i}{\prod_{k=1}^{i} (1 + r_k)} = \sum_{i=1}^{N} \frac{DV_i}{\prod_{k=i}^{N} (1 + r_k)}.$$

Indeed, the residual income model holds true under this simplest accounting system.

The same double-entry logic applies to cash flow accounting in which cash to be paid out and retained is the only stock account. In cash flow accounting, the entire flow transactions during the period are summarized as:

Cash $CF / Income $CF

while the only stock transaction is

Dividends $DV / Cash $DV.

Assuming the beginning stock amount of cash is $CS

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6 Cash to be retained is not recorded in this accounting system.
\[ BE_i = CS_i, \]

we obtain

\[ RI_i = NI_i - r_i BE_{i-1} = CF_i - r_i CS_{i-1} \]

and consequently

\[
VE_0 = BE_0 + \sum_{i=1}^{N} \frac{RI_i}{\prod_{k=1}^{i} (1 + r_k)} = CS_0 + \sum_{i=1}^{N} \frac{CF_i - r_i CS_{i-1}}{\prod_{k=1}^{i} (1 + r_k)}.
\]

From the thought experiments above, we may claim that dividends and cash flows are derived from net income and book value, not vice versa as usually regarded.

4. A Theory of Accounting Relativity

The dividend discount model is a special case of the residual income model and the fundamental value of the firm is a scalar invariant under any accounting system with the clean-surplus. Therefore, the estimate of the firm value is invariant once given a future sequence (including dividends series\(^7\)) in any system, as is pointed out by Lundholm and O’Keefe (2001).

However, if a sequence in a particular accounting system has simpler time series properties than another, say, AR (1) versus AR (10), the former could give us a more precise estimate of the firm value than the latter. Because this argument applies to a dividends sequence, it is likely that some accounting sequences lead to more precise estimates than dividends as well as other accounting series. The logic here may remind us of the fact that a transformation of coordinates makes a difficult problem more tractable in physics.

Although in principle any accounting system is relative in valuation, some systems are more decision-useful than others in the sense that the former give us more precise estimates than the

\(^7\) A dividends series is invariant under any accounting system.
latter. The decision-usefulness in this sense is unrelated to the so-called representative
faithfulness of accounting data to the market (current) value. Rather, as Fama and French (1992,
1993) and others have amply shown in empirical finance, the size of price-to-book ratio (PBR),
i.e., the divergence between book value and market value, is an important ingredient for asset
pricing⁸.

Ijiri (1975, 1981) continues to assert the importance of a historical cost-based accounting
system in our modern market society using hardness as the key concept from a
stewardship-oriented perspective. But my conclusion is that his claim is too modest. Not only
from a stewardship-oriented perspective but also from a decision-useful one, there is no
theoretical primacy of cash flow or current value accounting over historical cost accounting.

Under double-entry bookkeeping, any accounting system is relative concerning valuation in
the sense that the fundamental value is a scalar invariant. A dividend stream or any accounting
sequence has no a priori conceptual as well as empirical grounds to claim its primacy over others
in valuation.

⁸ There is a debate on why PBR matters in asset pricing. Davis et al. (2000) claim that it is due to
a market-wide risk factor which correlates with the PBR of each firm, while Daniel and Titman
(1997) claim that the market price of each firm is directly dependent on its PBR. However, what
is important is the fact that the divergence between book value and market value has some
information in asset pricing.
References


